Exercise Set 7.1*

1. Let $X = \{1, 3, 5\}$ and $Y = \{s, t, u, v\}$. Define $f : X \rightarrow Y$ by the following arrow diagram.

```
X  f  Y
1  f  1
3  f  t
5  f  u
```

a. Write the domain of $f$ and the co-domain of $f$.
b. Find $f(1)$, $f(3)$, and $f(5)$.
c. What is the range of $f$?
d. Is 3 an inverse image of $s$? Is 1 an inverse image of $u$?
e. What is the inverse image of $s$ of $u$? of $v$?
f. Represent $f$ as a set of ordered pairs.

2. Let $X = \{1, 3, 5\}$ and $Y = \{a, b, c, d\}$. Define $g : X \rightarrow Y$ by the following arrow diagram.

```
X  g  Y
1  g  a
3  g  b
5  g  c
```

a. Write the domain of $g$ and the co-domain of $g$.
b. Find $g(1)$, $g(3)$, and $g(5)$.
c. What is the range of $g$?
d. Is 3 an inverse image of $a$? Is 1 an inverse image of $b$?
e. What is the inverse image of $b$? of $c$?
f. Represent $g$ as a set of ordered pairs.

3. Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from $X$ to $Y$?

a. 

```
X  Y
2  2
4  4
5  6
```

b. 

```
X  Y
2  1
4  2
5  6
```

4. Indicate whether the statements in parts (a)-(d) are true or false. Justify your answers.

a. If two elements in the domain of a function are equal, then their images in the co-domain are equal.
b. If two elements in the co-domain of a function are equal, then their preimages in the domain are also equal.
c. A function can have the same output for more than one input.
d. A function can have the same input for more than one output.

5. a. Find all functions from $X = \{a, b\}$ to $Y = \{u, v\}$.
b. Find all functions from $X = \{a, b, c\}$ to $Y = \{u\}$.
c. Find all functions from $X = \{a, b, c\}$ to $Y = \{u, v\}$.

6. a. How many functions are there from a set with three elements to a set with four elements?
b. How many functions are there from a set with five elements to a set with two elements?
c. How many functions are there from a set with $m$ elements to a set with $n$ elements, where $m$ and $n$ are positive integers?

7. Define functions $f$ and $g$ from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas:

For all $x \in \mathbb{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Show that $f = g$.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol $\ast$ indicates that only a hint or a partial solution is given. The symbol $\ast$ signals that an exercise is more challenging than usual.
8. Define functions $H$ and $K$ from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas:
   
   \[ H(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad K(x) = \lfloor x \rfloor. \]
   
   Does $H = K$? Explain.

9. Let $F$ and $G$ be functions from the set of all real numbers to itself. Define the product functions $F \cdot G: \mathbb{R} \to \mathbb{R}$ and $G \cdot F: \mathbb{R} \to \mathbb{R}$ as follows:
   
   \[ (F \cdot G)(x) = F(x) \cdot G(x) \quad \text{for all } x \in \mathbb{R}, \]
   
   \[ (G \cdot F)(x) = G(x) \cdot F(x) \quad \text{for all } x \in \mathbb{R}. \]
   
   Does $F \cdot G = G \cdot F$? Explain.

10. Let $F$ and $G$ be functions from the set of all real numbers to itself. Define new functions $F - G: \mathbb{R} \to \mathbb{R}$ and $G - F: \mathbb{R} \to \mathbb{R}$ as follows:
   
   \[ (F - G)(x) = F(x) - G(x) \quad \text{for all } x \in \mathbb{R}, \]
   
   \[ (G - F)(x) = G(x) - F(x) \quad \text{for all } x \in \mathbb{R}. \]
   

11. Let $\mathbb{Z}$ be the identity function defined on the set of all integers, and suppose that $e, b^j, K(t)$, and $u_{ij}$ all represent integers. Find
   
   a. $i_2(e)$  
   b. $i_2(b^j)$  
   c. $i_2(K(t))$  
   d. $i_2(u_{ij})$

12. Find functions defined on the set of nonnegative integers that define the sequences whose first six terms are given below.
   
   a. $1, -\frac{1}{2}, 1, \frac{1}{2}, 1, -\frac{1}{2}$  
   b. $0, -2, 4, -6, 8, -10$

13. Let $A = \{1, 2, 3, 4, 5\}$ and define a function $F: \mathcal{P}(A) \to \mathbb{Z}$ as follows: For all sets $X$ in $\mathcal{P}(A)$,
   
   \[ F(X) = \begin{cases} 0 & \text{if } X \text{ has an even number of elements} \\ 1 & \text{if } X \text{ has an odd number of elements} \end{cases} \]
   
   Find the following:
   
   a. $F(\{1, 3, 4\})$  
   b. $F(\emptyset)$  
   c. $F([2, 3])$  
   d. $F([2, 3, 4, 5])$

14. Let $S$ be the set of all strings of $a$'s and $b$'s. a. Define $f: S \to \mathbb{Z}$ as follows: For each string $s$ in $S$,
   
   \[ f(s) = \begin{cases} \text{the number of } b \text{'s to the left of the left-most } a \text{ in } s & \text{if } s \text{ contains no } a \text{'s} \\ 0 & \text{if } s \text{ contains no } a \text{'s} \end{cases} \]
   
   Find $f(aba), f(bbab)$, and $f(b)$. What is the range of $f$?

   b. Define $g: S \to S$ as follows: For each string $s$ in $S$,
   
   \[ g(s) = \text{the string obtained by writing the characters of } s \text{ in reverse order}. \]
   
   Find $g(aba), g(bbab)$, and $g(b)$. What is the range of $g$?

15. Use the definition of logarithm to fill in the blanks below.
   
   a. $\log_2 8 = 3$ because $2^3 = 8$.
   b. $\log_3 (\frac{1}{3}) = -1$ because $3^{-1} = \frac{1}{3}$.
   c. $\log_4 16 = 2$ because $4^2 = 16$.
   d. $\log_3 1 = 0$ because $3^0 = 1$.
   e. $\log_2 1 = 0$ because $2^0 = 1$.

16. Find exact values for each of the following quantities. Do not use a calculator.
   
   a. $\log_2 8$  
   b. $\log_3 1024$  
   c. $\log_9 (\frac{1}{9})$  
   d. $\log_2 1$  
   e. $\log_{10} (\frac{1}{10})$  
   f. $\log_3 3$

   g. $\log_2 (2^3)$

17. Use the definition of logarithm to prove that for any positive real number $b$ with $b \neq 1$, $\log_b b = 1$.

18. Use the definition of logarithm to prove that for any positive real number $b$ with $b \neq 1$, $\log_b 1 = 0$.

19. If $h$ is any positive real number and $x$ is any real number, $b^x$ is defined as follows: $b^x = \frac{1}{h^x}$. Use this definition and the definition of logarithm to prove that $\log_h \left( \frac{1}{h} \right) = -\log_h (h)$ for all positive real numbers $h$.

20. Use the unique factorization theorem (Section 3.3) and the definition of logarithm to prove that $\log_2 7$ is irrational.

21. If $h$ and $y$ are positive real numbers such that $\log_h y = 3$, what is $\log_3 (y)$? Why?

22. If $h$ and $y$ are positive real numbers such that $\log_h y = 2$, what is $\log_2 (y)$? Why?

23. Let $A = \{2, 3, 5\}$ and $B = \{x, y\}$. Let $p_1$ and $p_2$ be the projections of $A \times B$ onto the first and second coordinates. That is, for each pair $(a, b) \in A \times B$, $p_1(a, b) = a$ and $p_2(a, b) = b$.
   
   a. Find $p_1(2, y)$ and $p_2(5, x)$. What is the range of $p_1$?
   
   b. Find $p_1(2, y)$ and $p_2(5, x)$. What is the range of $p_2$?

24. Observe that $\text{mod}$ and $\text{div}$ can be defined as functions from $\mathbb{Z} \times \mathbb{N}$ to $\mathbb{Z}$. For each ordered pair $(a, d)$ consisting of a nonnegative integer $a$ and a positive integer $d$, let
   
   \[ \text{mod}(a, d) = a \text{ mod } d \quad \text{(the nonnegative remainder obtained when } a \text{ is divided by } d) \]
   
   \[ \text{div}(a, d) = a \text{ div } d \quad \text{(the integer quotient obtained when } a \text{ is divided by } d). \]
   
   Find each of the following:
   
   a. $\text{mod}(67, 10)$ and $\text{div}(67, 10)$
   b. $\text{mod}(59, 8)$ and $\text{div}(59, 8)$
   c. $\text{mod}(30, 5)$ and $\text{div}(30, 5)$

25. Consider the coding and decoding functions $E$ and $D$ defined in Example 7.1.10.
   
   a. Find $E(0110)$ and $D(11111000111111)$. 
   b. Find $E(10100)$ and $D(00000011111111)$. 

26. Consider the Hamming distance function defined in Example 7.1.11.
   
   a. Find $H(101010, 0000111)$. 
   b. Find $H(00110, 10111)$. 

27. Consider the following function $g$:
   
   \[ g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \]
   
   a. Find $g(-2)$.
   b. Find $g(-3)$.
   c. Find $g(0)$.
   d. Find $g(1)$.
27. A permutation on a set can be regarded as a function from the set to itself. For instance, one permutation of \(\{1, 2, 3, 4\}\) is 2341. It can be identified with the function that sends each position number to the number occupying that position. Since position 1 is occupied by 2, 1 is sent to 2 or 1 → 2; since position 2 is occupied by 3, 2 is sent to 3 or 2 → 3; and so forth. The entire permutation can be written using arrows as follows:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
2 & 3 & 4 & 1 \\
\end{array}
\]

a. Use arrows to write each of the six permutations of \(\{1, 2, 3\}\).
b. Use arrows to write each of the permutations of \(\{1, 2, 3, 4\}\) that keep 2 and 4 fixed.
c. Which permutations of \(\{1, 2, 3\}\) keep no elements fixed?
d. Use arrows to write all permutations of \(\{1, 2, 3, 4\}\) that keep no elements fixed.

28. Draw arrow diagrams for the Boolean functions defined by the following input/output tables.

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>(Q)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

30. Consider the three-place Boolean function \(f\) defined by the following rule: For each triple \((x_1, x_2, x_3)\) of 0's and 1's,

\[
f(x_1, x_2, x_3) = (4x_1 + 3x_2 + 2x_3) \mod 2.
\]

a. Find \(f(1, 1, 1)\) and \(f(0, 0, 1)\).
b. Describe \(f\) using an input/output table.

31. Student A tries to define a function \(g: \mathbb{Q} \to \mathbb{Z}\) by the rule

\[
g\left(\frac{m}{n}\right) = m - n, \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.
\]

Student B claims that \(g\) is not well defined. Justify student B's claim.

32. Student C tries to define a function \(h: \mathbb{Q} \to \mathbb{Q}\) by the rule

\[
h\left(\frac{m}{n}\right) = \frac{m^2}{n}, \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.
\]

Student D claims that \(h\) is not well defined. Justify student D's claim.

33. On certain computers the integer data type goes from \(-2,147,483,648\) through \(2,147,483,647\). Let \(S\) be the set of all integers from \(-2,147,483,648\) through \(2,147,483,647\). Try to define a function \(f: S \to S\) by the rule \(f(n) = n^2\) for each \(n\) in \(S\). Is \(f\) well defined? Why?

34. Given a set \(S\) and a subset \(A\), the characteristic function of \(A\), denoted \(\chi_A\), is the function defined from \(S\) to \(\mathbb{Z}\) with the property that for all \(u \in S\),

\[
\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}
\]

Show that each of the following holds for all subsets \(A\) and \(B\) of \(S\) and all \(u \in S\).

a. \(\chi_{A \cup B}(u) = \chi_A(u) \cdot \chi_B(u)\)
b. \(\chi_{A \cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)\)

each of exercises 35–39 refers to the Euler phi function, denoted \(\phi\), which is defined as follows: For each integer \(n \geq 1\), \(\phi(n)\) is the number of positive integers less than or equal to \(n\) that have no common factors with \(n\) except ±1. For example, \(\phi(10) = 4\) because there are four positive integers less than or equal to 10 that have no common factors with 10 except ±1; namely, 1, 3, 7, and 9.

35. Find each of the following:

a. \(\phi(15)\)
b. \(\phi(2)\)
c. \(\phi(5)\)
d. \(\phi(12)\)
e. \(\phi(11)\)
f. \(\phi(1)\)

36. Prove that if \(p\) is a prime number and \(n\) is an integer with \(n \geq 1\), then \(\phi(n^p) = n^p - n^{p-1}\).

37. Prove that there are infinitely many integers \(n\) for which \(\phi(n)\) is a perfect square.

38. Use the inclusion/exclusion principle to prove the following: If \(n = pq\), where \(p\) and \(q\) are distinct prime numbers, then \(\phi(n) = (p - 1)(q - 1)\).
Exercise Set 7.2

1. The definition of one-to-one is stated in two ways:
\[ \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2 \]

and
\[ \forall x_1, x_2 \in X, \text{ if } x_1 \neq x_2 \text{ then } F(x_1) \neq F(x_2). \]

Why are these two statements logically equivalent?

2. Fill in the blanks with the word most or least.
   a. A function \( F \) is one-to-one if, and only if, each element in the co-domain of \( F \) is the image of at ______ one element in the domain of \( F \).
   b. A function \( F \) is onto if, and only if, each element in the co-domain of \( F \) is the image of at ______ one element in the domain of \( F \).

3. When asked to state the definition of one-to-one, a student replies, "A function \( f \) is one-to-one if, and only if, every element of \( X \) is sent by \( f \) to exactly one element of \( Y \). Give a counterexample to show that the student's reply is incorrect.

4. Let \( f: X \to Y \) be a function. True or false? A sufficient condition for \( f \) to be one-to-one is that for all elements \( y \) in \( Y \), there is at most one \( x \) in \( X \) with \( f(x) = y \).

5. All but two of the following statements are correct ways to express the fact that a function \( f \) is onto. Find the two that are incorrect.
   a. \( f \) is onto \( \Leftrightarrow \) every element in its co-domain is the image of some element in its domain.
   b. \( f \) is onto \( \Leftrightarrow \) every element in its domain has a corresponding image in its co-domain.
   c. \( f \) is onto \( \Leftrightarrow \forall y \in Y, \exists x \in X \text{ such that } f(x) = y. \)
   d. \( f \) is onto \( \Leftrightarrow \forall x \in X, \exists y \in Y \text{ such that } f(x) = y. \)
   e. \( f \) is onto \( \Leftrightarrow \) the range of \( f \) is the same as the co-domain of \( f \).

6. Let \( X = \{1, 5, 9\} \) and \( Y = \{3, 4, 7\} \).
   a. Define \( f: X \to Y \) by specifying that 
      \[ f(1) = 4, \quad f(5) = 7, \quad f(9) = 4. \]
      Is \( f \) one-to-one? Is \( f \) onto? Explain your answers.
   b. Define \( g: X \to Y \) by specifying that 
      \[ g(1) = 7, \quad g(5) = 3, \quad g(9) = 4. \]
      Is \( g \) one-to-one? Is \( g \) onto? Explain your answers.

7. Let \( X = \{a, b, c, d\} \) and \( Y = \{x, y, z\} \). Define functions \( F \) and \( G \) by the arrow diagrams below.

   ![Diagram of Domain and Co-domain of F and G]

   a. Is \( F \) one-to-one? Why or why not? Is it onto? Why or why not?
   b. Is \( G \) one-to-one? Why or why not? Is it onto? Why or why not?

8. Let \( X = \{a, b, c\} \) and \( Y = \{w, x, y, z\} \). Define functions \( H \) and \( K \) by the arrow diagrams below.

   ![Diagram of Domain and Co-domain of H and K]

   a. Is \( H \) one-to-one? Why or why not? Is it onto? Why or why not?
   b. Is \( K \) one-to-one? Why or why not? Is it onto? Why or why not?
Let \( X = \{1, 2, 3\} \), \( Y = \{1, 2, 3, 4\} \), and \( Z = \{1, 2\} \).

a. Define a function \( f: X \rightarrow Y \) that is one-to-one but not onto.

d. Define a function \( k: X \rightarrow X \) that is one-to-one and onto, but is not the identity function on \( X \).

10. a. How many one-to-one functions are there from a set with three elements to a set with four elements?

b. How many one-to-one functions are there from a set with three elements to a set with two elements?

c. How many one-to-one functions are there from a set with three elements to a set with three elements?

d. How many one-to-one functions are there from a set with three elements to a set with five elements?

11. a. How many onto functions are there from a set with three elements to a set with two elements?

b. How many onto functions are there from a set with three elements to a set with five elements?

c. How many onto functions are there from a set with three elements to a set with three elements?

d. How many onto functions are there from a set with four elements to a set with two elements?

e. How many onto functions are there from a set with four elements to a set with three elements?

12. a. Define \( f: Z \rightarrow Z \) by the rule \( f(n) = 2n \), for all integers \( n \).

(i) Is \( f \) one-to-one? Prove or give a counterexample.

(ii) Is \( f \) onto? Prove or give a counterexample.

b. Define \( g: Z \rightarrow Z \) by the rule \( g(n) = 4n - 5 \), for all integers \( n \).

(i) Is \( g \) one-to-one? Prove or give a counterexample.

(ii) Is \( g \) onto? Prove or give a counterexample.

13. a. Define \( h: Z \rightarrow Z \) by the rule \( h(n) = 4n + 3 \), for all integers \( n \).

(i) Is \( h \) one-to-one? Prove or give a counterexample.

(ii) Is \( h \) onto? Prove or give a counterexample.

b. Define \( G: R \rightarrow R \) by the rule \( G(x) = 4x - 5 \), for all real numbers \( x \).

Is \( G \) onto? Prove or give a counterexample.

14. a. Define \( H: R \rightarrow R \) by the rule \( H(x) = x^2 \), for all real numbers \( x \).

(i) Is \( H \) one-to-one? Prove or give a counterexample.

(ii) Is \( H \) onto? Prove or give a counterexample.

b. Define \( K: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) by the rule \( K(x) = x^2 \), for all nonnegative real numbers \( x \).

Is \( K \) onto? Prove or give a counterexample.

15. Explain the mistake in the following "proof."

**Theorem:** The function \( f: Z \rightarrow Z \) defined by the formula \( f(n) = 4n + 3 \), for all integers \( n \), is one-to-one.

**Proof:** Suppose any integer \( n \) is given. Then by definition of \( f \), there is only one possible value for \( f(n) \), namely, \( 4n + 3 \). Hence \( f \) is one-to-one.

In each of 16–19 a function \( f \) is defined on a set of real numbers. Determine whether or not \( f \) is one-to-one and justify your answer.

16. \( f(x) = \frac{x + 1}{x} \), for all real numbers \( x \neq 0 \)

17. \( f(x) = \frac{x}{x^2 + 1} \), for all real numbers \( x \)

18. \( f(x) = \frac{3x - 1}{x} \), for all real numbers \( x \neq 0 \)

19. \( f(x) = \frac{x + 1}{x - 1} \), for all real numbers \( x \neq 1 \)

20. Referring to Example 7.2.3. assume that records with the following social security numbers are to be placed in sequence into Table 7.2.1. Find the position into which each record is placed.

   a. 417-30-2072
   b. 364-98-1703
   c. 283-09-0787

21. Define Floor: \( R \rightarrow Z \) by the formula \( \text{Floor}(x) = \lfloor x \rfloor \), for all real numbers \( x \).

   a. Is Floor one-to-one? Prove or give a counterexample.
   b. Is Floor onto? Prove or give a counterexample.

22. Let \( S \) be the set of all strings of 0's and 1's, and define \( f: S \rightarrow \mathbb{Z}_{\geq 0} \) by

   \[ f(s) = \text{the length of } s \quad \text{for all strings } s \text{ in } S. \]

   a. Is \( f \) one-to-one? Prove or give a counterexample.
   b. Is \( f \) onto? Prove or give a counterexample.

23. Let \( S \) be the set of all strings of 0's and 1's, and define \( D: S \rightarrow \mathbb{Z} \) as follows: For all \( s \in S \),

   \[ D(s) = \text{the number of 1's in } s \text{ minus the number of 0's in } s. \]

   a. Is \( D \) one-to-one? Prove or give a counterexample.
   b. Is \( D \) onto? Prove or give a counterexample.

24. Define \( F: \mathcal{P}([a, b, c]) \rightarrow Z \) as follows: For all \( A \) in \( \mathcal{P}([a, b, c]) \),

   \[ F(A) = \text{the number of elements in } A. \]

   a. Is \( F \) one-to-one? Prove or give a counterexample.
   b. Is \( F \) onto? Prove or give a counterexample.

25. Let \( S \) be the set of all strings of 0's and 1's, and define \( N: S \rightarrow Z \) by

   \[ N(s) = \text{the number of 0's in } s \quad \text{for all } s \in S. \]

   a. Is \( N \) one-to-one? Prove or give a counterexample.
   b. Is \( N \) onto? Prove or give a counterexample.
Let $S$ be the set of all strings in $a$'s and $b$'s, and define $C: S \to S$ by

$$C(s) = as, \quad \text{for all } s \in S.$$  

$C$ is called concatenation by $a$ on the left.

(a) Is $C$ one-to-one? Prove or give a counterexample.

(b) Is $C$ onto? Prove or give a counterexample.

#27. Define $F: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and $G: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows: for all $(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$,

$$F(n, m) = 3^n 5^m \quad \text{and} \quad G(n, m) = 3^n 6^m.$$  

(a) Is $F$ one-to-one? Prove or give a counterexample.

(b) Is $G$ one-to-one? Prove or give a counterexample.

28. a. Is $\log_a 27 = \log_3 3$? Why or why not?

b. Is $\log_{10} 9 = \log_3 3$? Why or why not?

The properties of logarithms established in 29 and 30 are used in Sections 9.4 and 9.5.

29. Prove that for all positive real numbers $b$, $x$, and $y$ with $b \neq 1$,

$$\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y.$$  

30. Prove that for all positive real numbers $b$, $x$, and $y$ with $b \neq 1$,

$$\log_b (xy) = \log_b x + \log_b y.$$  

31. Prove that for all real numbers $a$, $b$, and $x$ with $b$ and $x$ positive and $b \neq 1$,

$$\log_b (x^n) = n \log_b x.$$  

Exercises 32 and 33 use the following definition: If $f: R \to R$ and $g: R \to R$ are functions, then the function $(f + g): R \to R$ is defined by the formula $(f + g)(x) = f(x) + g(x)$ for all real numbers $x$.

32. If $f: R \to R$ and $g: R \to R$ are both one-to-one, is $f + g$ also one-to-one? Justify your answer.

33. If $f: R \to R$ and $g: R \to R$ are both onto, is $f + g$ also onto? Justify your answer.

Exercises 34 and 35 use the following definition: If $f: R \to R$ is a function and $c$ is a nonzero real number, the function $(c \cdot f): R \to R$ is defined by the formula $(c \cdot f)(x) = c \cdot f(x)$ for all real numbers $x$.

34. Let $f: R \to R$ be a function and $c$ a nonzero real number.

(a) If $f$ is one-to-one, is $c \cdot f$ also one-to-one? Justify your answer.

(b) If $f$ is onto, is $c \cdot f$ also onto? Justify your answer.

Let $X = \{a, b, c, d, e\}$ and $Y = \{s, t, u, v, w\}$. In each of 36 and 37 a one-to-one correspondence $F: X \to Y$ is defined by an arrow diagram. In each case draw an arrow diagram for $F^{-1}$.

36.

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (0,-1) {$b$};
  \node (c) at (0,-2) {$c$};
  \node (d) at (0,-3) {$d$};
  \node (e) at (0,-4) {$e$};
  \node (s) at (1,0) {$s$};
  \node (t) at (1,-1) {$t$};
  \node (u) at (1,-2) {$u$};
  \node (v) at (1,-3) {$v$};
  \node (w) at (1,-4) {$w$};

  \draw[->] (a) -- (s);
  \draw[->] (b) -- (t);
  \draw[->] (c) -- (u);
  \draw[->] (d) -- (v);
  \draw[->] (e) -- (w);

  \node at (0,-5) {$X$};
  \node at (1,-5) {$Y$};
\end{tikzpicture}
\end{center}

37.

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (0,-1) {$b$};
  \node (c) at (0,-2) {$c$};
  \node (d) at (0,-3) {$d$};
  \node (e) at (0,-4) {$e$};
  \node (s) at (1,0) {$s$};
  \node (t) at (1,-1) {$t$};
  \node (u) at (1,-2) {$u$};
  \node (v) at (1,-3) {$v$};
  \node (w) at (1,-4) {$w$};

  \draw[->] (a) -- (s);
  \draw[->] (b) -- (t);
  \draw[->] (c) -- (u);
  \draw[->] (d) -- (v);
  \draw[->] (e) -- (w);

  \node at (0,-5) {$X$};
  \node at (1,-5) {$Y$};
\end{tikzpicture}
\end{center}

In 38–51 indicate which of the functions in the referenced exercise are one-to-one correspondences. For each function that is a one-to-one correspondence, find the inverse function.

38. Exercise 12a

39. Exercise 12b

40. Exercise 13a

41. Exercise 13b

42. Exercise 14a

43. Exercise 21

44. Exercise 22

45. Exercise 23

46. Exercise 24

47. Exercise 25

48. Exercise 16 with the co-domain taken to be the set of all real numbers not equal to 1.

H 49. Exercise 17 with the co-domain taken to be the set of all real numbers.

50. Exercise 18 with the co-domain taken to be the set of all real numbers not equal to 3.

51. Exercise 19 with the co-domain taken to be the set of all real numbers not equal to 1.

52. In Example 7.2.8 a one-to-one correspondence was defined from the power set of $\{a, b\}$ to the set of all strings of 0's and 1's that have length 2. Thus the elements of these two sets can be matched up exactly, and so the two sets have the same number of elements.

(a) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set with $n$ elements. Use Example 7.2.8 as a model to define a one-to-one correspondence from $\mathcal{P}(X)$, the set of all subsets of $X$, to the set of all strings of 0's and 1's that have length $n$.

(b) Use the one-to-one correspondence of part (a) to deduce that a set with $n$ elements has $2^n$ subsets. (This provides an alternative proof of Theorem 5.3.5.)

H 53. Write a computer algorithm to check whether a function from one finite set to another is one-to-one. Assume the existence of an independent algorithm to compute values of the function.

H 54. Write a computer algorithm to check whether a function from one finite set to another is onto. Assume the existence of an independent algorithm to compute values of the function.
Exercise Set 7.3

1. a. If 4 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?
   b. If 5 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?
2. a. If 13 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?
   b. If 20 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?
3. A small town has only 500 residents. Must there be 2 residents who have the same birthday? Why?
4. In a group of 700 people, must there be 2 who have the same first and last initials? Why?
5. a. Given any set of four integers, must there be two that have the same remainder when divided by 3? Why?
   b. Given any set of three integers, must there be two that have the same remainder when divided by 3? Why?
6. a. Given any set of seven integers, must there be two that have the same remainder when divided by 6? Why?
   b. Given any set of seven integers, must there be two that have the same remainder when divided by 8? Why?
7. Let \( S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \). Suppose six integers are chosen from \( S \). Must there be two integers whose sum is 15? Why?
8. Let \( T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Suppose five integers are chosen from \( T \). Must there be two integers whose sum is 10? Why?
9. a. If seven integers are chosen from between 1 and 12 inclusive, must at least one of them be odd? Why?
   b. If ten integers are chosen from between 1 and 20 inclusive, must at least one of them be even? Why?
10. If \( n + 1 \) integers are chosen from the set
   \[ \{1, 2, 3, \ldots, 2n\}, \]
   where \( n \) is a positive integer, must at least one of them be odd? Why?
11. If \( n + 1 \) integers are chosen from the set
   \[ \{1, 2, 3, \ldots, 2n\}, \]
   where \( n \) is a positive integer, must at least one of them be even? Why?
12. How many cards must you pick from a standard 52-card deck to be sure of getting at least 1 red card? Why?
13. Suppose six pairs of similar-looking boots are thrown together in a pile. How many individual boots must you pick to be sure of getting a matched pair? Why?
14. How many integers from 0 through 60 must you pick in order to be sure of getting at least one that is odd? at least one that is even?
15. If \( n \) is a positive integer, how many integers from 0 through \( 2n \) must you pick in order to be sure of getting at least one that is odd? at least one that is even?
16. How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?
17. How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 7?
18. How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 15?
19. How many integers from 100 through 999 must you pick in order to be sure that at least two of them have a digit in common? (For example, 256 and 530 have the common digit 5.)
20. If repeated divisions by 20,483 are performed, how many distinct remainders can be obtained?
21. When \( 5/20483 \) is written as a decimal, what is the maximum length of the repeating section of the representation?
22. Is 0.1010010001000001... (where each string of 0's is one longer than the previous one) rational or irrational?
23. Is 0.56, 566555666555566666... (where the strings of 5's and 6's become longer in each repetition) rational or irrational?
24. Show that within any set of thirteen integers chosen from 2 through 40, there are at least two integers with a common divisor greater than 1.
25. In a group of 30 people, must at least 3 have been born in the same month? Why?
26. In a group of 30 people, must at least 4 have been born in the same month? Why?
27. In a group of 2,000 people, must at least 5 have the same birthday? Why?
28. A programmer writes 500 lines of computer code in 17 days. Must there have been at least 1 day when the programmer wrote 30 or more lines of code? Why?
29. A certain college class has 40 students. All the students in the class are known to be from 17 through 34 years of age. You want to make a bet that the class contains at least \( x \) students of the same age. How large can you make \( x \) and yet be sure to win your bet?
30. A penny collection contains twelve 1967 pennies, seven 1968 pennies, and eleven 1971 pennies. If you are to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least five pennies from the same year?
Exercise Set 7.4

In each of 1 and 2, functions \( f \) and \( g \) are defined by arrow diagrams. Find \( g \circ f \) and \( f \circ g \) and determine whether \( g \circ f \) equals \( f \circ g \).

1. \[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
3 & \rightarrow & 3 \\
5 & \rightarrow & 5 \\
\end{array}
\]

2. \[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
3 & \rightarrow & 3 \\
5 & \rightarrow & 5 \\
\end{array}
\]

In each of 3–6, functions \( F \) and \( G \) are defined by formulas. Find \( G \circ F \) and \( F \circ G \) and determine whether \( G \circ F \) equals \( F \circ G \).

3. \( F(x) = x^3 \) and \( G(x) = x - 1 \), for all real numbers \( x \).

4. \( F(x) = x^5 \) and \( G(x) = x^{1/5} \) for all real numbers \( x \).

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
3 & \rightarrow & 3 \\
5 & \rightarrow & 5 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
3 & \rightarrow & 3 \\
5 & \rightarrow & 5 \\
\end{array}
\]