Chapter 8

Combinatorial Analysis

8.1 FUNDAMENTAL PRINCIPLE OF COUNTING

Combinatorial analysis, which includes the study of permutations, combinations and partitions, is concerned with determining the number of logical possibilities of some event without necessarily enumerating each case. The following basic principle of counting is used throughout.

Fundamental Principle of Counting: If some event can occur in \( n_1 \) different ways, and if, following this event, a second event can occur in \( n_2 \) different ways, and, following this second event, a third event can occur in \( n_3 \) different ways, \ldots, then the number of ways the events can occur in the order indicated is \( n_1 \cdot n_2 \cdot n_3 \cdot \ldots \).

**EXAMPLE 8.1.**

(a) Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

Each letter can be printed in twenty-six different ways, the first digit in nine ways and each of the other two digits in ten ways. Hence

\[
26 \cdot 26 \cdot 9 \cdot 10 \cdot 10 = 608,400
\]

different plates can be printed.

(b) In how many ways can an organization containing twenty-six members elect a president, treasurer and secretary (assuming no person is elected to more than one position)?

The president can be elected in twenty-six different ways; following this, the treasurer can be elected in twenty-five different ways (since the person chosen president is not eligible to be treasurer); and, following this, the secretary can be elected in twenty-four different ways. Thus, by the above principle of counting, there are

\[
26 \cdot 25 \cdot 24 = 15,600
\]
different ways in which the organization can elect the officers.

8.2 FACTORIAL NOTATION

The product of the positive integers from 1 to \( n \) inclusive is denoted by \( n! \) (read “\( n \) factorial”):

\[
n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n
\]

In other words, \( n! \) is defined by

\[
1! = 1 \quad \text{and} \quad n! = n \cdot (n-1)!
\]

It is also convenient to define \( 0! = 1 \).

**EXAMPLE 8.2.**

(a) \( 2! = 1 \cdot 2 = 2 \quad 3! = 1 \cdot 2 \cdot 3 = 6 \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \)

\[
5! = 5 \cdot 4! = 5 \cdot 24 = 120 \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720
\]

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\( 5! = 5 \cdot 4! = 5 \cdot 24 = 120 \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720 \)
8.3 BINOMIAL COEFFICIENTS

The symbol \( \binom{n}{r} \) (read "nCr"), where \( r \) and \( n \) are positive integers with \( r \leq n \), is defined as follows

\[
\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r}
\]

By Example 8.2(c), we see that

\[
\binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} = \frac{n!}{(n-r)!r!}
\]

But \( n - (n-r) = r \); hence we have the following important relation:

\[
\binom{n}{n-r} = \binom{n}{r} \quad \text{or, in other words, if } a + b = n \text{ then } \binom{n}{a} = \binom{n}{b}
\]

**EXAMPLE 8.3.**

(a) \( \binom{8}{2} = \binom{9}{4} = \binom{10}{3} = \frac{8 \cdot 7}{1 \cdot 2} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792 \)

(b) \( \binom{18}{1} = \frac{18}{1} = 18 \)

Note that \( \binom{n}{r} \) has exactly \( r \) factors in both the numerator and the denominator.

(b) Compute \( \binom{10}{7} \). By definition,

\[
\binom{10}{7} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 120
\]

On the other hand, \( 10 - 7 = 3 \) and so we can also compute \( \binom{10}{7} \) as follows:

\[
\binom{10}{7} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120
\]

Observe that the second method saves space and time.

The numbers \( \binom{n}{r} \) are called the binomial coefficients since they appear as the coefficients in the expansion of \((a + b)^n\). Specifically, one can prove that

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]
8.3 BINOMIAL COEFFICIENTS

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\[
\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r}
\]

By Example 8.2(c), we see that

\[
\binom{n}{r} = \frac{n(n-1)(n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} \frac{n!}{r!(n-r)!}
\]

But \( n-(n-r)=r \); hence we have the following important relation:

\[
\binom{n}{n-r} = \binom{n}{r} \text{ or, in other words, if } a+b=n \text{ then } \binom{n}{a} = \binom{n}{b}
\]

**EXAMPLE 8.3.**

(a) \( \binom{8}{2} = \frac{8 \cdot 7}{2} = 28 \) \( \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126 \) \( \binom{12}{5} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792 \)

(b) Compute \( \binom{10}{7} \). By definition,

\[
\binom{10}{7} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 120
\]

On the other hand, \( 10-7=3 \) and so we can also compute \( \binom{10}{7} \) as follows:

\[
\binom{10}{7} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120
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Observe that the second method saves space and time.

The numbers \( \binom{n}{r} \) are called the binomial coefficients since they appear as the coefficients in the expansion of \( (a+b)^n \). Specifically, one can prove that

\[
(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]
The coefficients of the successive powers of \(a + b\) can be arranged in a triangular array of numbers, called Pascal’s triangle, as follows:

\[
\begin{align*}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a + b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{align*}
\]

Pascal’s triangle has the following interesting properties.

(i) The first number and the last number in each row is 1.

(ii) Every other number in the array can be obtained by adding the two numbers appearing directly above it. For example, 10 = 4 + 6, 15 = 5 + 10, 20 = 10 + 10.

Since the numbers appearing in Pascal’s triangle are the binomial coefficients, property (ii) of Pascal’s triangle comes from the following theorem (proved in Problem 8.8).

**Theorem 8.1:**

\[
\binom{n + 1}{r} = \left( \binom{n}{r - 1} \right) + \binom{n}{r}
\]

### 8.4 PERMUTATIONS

Any arrangement of a set of \(n\) objects in a given order is called a permutation of the objects (taken all at a time). Any arrangement of any \(r = n\) of these objects in a given order is called an \(r\)-permutation or a permutation of the \(n\) objects taken \(r\) at a time. Consider, for example, the set of letters \(a, b, c\) and \(d\). Then:

(i) \(bdca, dcba\) and \(acdb\) are permutations of the four letters (taken all at a time);

(ii) \(bad, adb, cdb\) and \(bca\) are permutations of the four letters taken three at a time;

(iii) \(ad, cb, da\) and \(bd\) are permutations of the four letters taken two at a time.

The number of permutations of \(n\) objects taken \(r\) at a time is denoted by

\[
P(n, r) = P_r, P_{n,r}, P_r^n, \text{ or } \binom{n}{r}.
\]

We shall use \(P(n, r)\). Before we derive the general formula for \(P(n, r)\) we consider a particular case.

**Example 8.4.** How many permutations are there of six objects, say \(a, b, c, d, e\), and \(f\), taken three at a time? In other words, we want to find the number of “three-letter words” using the above six letters without repetitions.

Let the general three-letter word be represented by the following three boxes:

\[
\_
\ , \,
\]

Now the first letter can be chosen in six different ways; following this, the second letter can be chosen in five different ways; and, following this, the last letter can be chosen in four different ways. Write each number in its appropriate box as follows:

\[
6 \, 5 \, 4
\]
Thus by the fundamental principle of counting there are $6 \cdot 5 \cdot 4 = 120$ possible three-letter words without repetitions from the six letters, or there are 120 permutations of six objects taken three at a time:

$$P(6,3) = 120$$

The derivation of the formula for the number of permutations of $n$ objects taken $r$ at a time, or the number of $r$-permutations of $n$ objects, $P(n,r)$, follows the procedure in the preceding example. The first element in an $r$-permutation of $n$ objects can be chosen in $n$ different ways; following this, the second element in the permutation can be chosen in $n-1$ ways; and, following this, the third element in the permutation can be chosen in $n-2$ ways. Continuing in this manner, we have that the $r$th (last) element in the $r$-permutation can be chosen in $n-(r-1) = n-r+1$ ways. Thus, by the fundamental principle of counting, we have

$$P(n,r) = n(n-1)(n-2) \cdots (n-r+1)$$

By Example 8.2(c), we see that

$$n(n-1)(n-2) \cdots (n-r+1) = \frac{n(n-1)(n-2) \cdots (n-r+1) \cdot (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Thus we have proven

**Theorem 8.2:** $P(n,r) = \frac{n!}{(n-r)!}$

In the special case in which $r = n$, we have

$$P(n,n) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

Accordingly,

**Corollary 8.3:** There are $n!$ permutations of $n$ objects (taken all at a time).

For example, there are $3! = 1 \cdot 2 \cdot 3 = 6$ permutations of the three letters $a$, $b$ and $c$. These are $abc$, $acb$, $bca$, $bca$, $cab$, $cba$.

### 8.5 PERMUTATIONS AND REPETITIONS

Frequently we want to find the number of permutations of objects some of which are alike, as illustrated in the examples below. The general formula is as follows:

**Theorem 8.4:** The number of permutations of $n$ objects of which $n_1$ are alike, $n_2$ are alike, ..., $n_r$ are alike is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

We indicate the proof of the above theorem by a particular example. Suppose we want to form all possible five-letter "words" using the letters from the word "DADDY". Now there are $5! = 120$ permutations of the objects $D_1, A, D_2, D_3, Y$, where the three $D$'s are distinguished. Observe that the following six permutations

$D_1D_2D_3AY$, $D_2D_1D_3AY$, $D_3D_1D_2AY$, $D_1D_3D_2AY$, $D_2D_3D_1AY$, and $D_3D_2D_1AY$

produce the same word when the subscripts are removed. The 6 comes from the fact that there are $3! = 3 \cdot 2 \cdot 1 = 6$ different ways of placing the three $D$'s in the first three positions in the permutation. This is true for each set of three positions in which the $D$'s can appear. Accordingly there are

$$\frac{5!}{3!} = \frac{120}{6} = 20$$

different five-letter words that can be formed using the letters from the word "DADDY".
EXAMPLE 8.5.

(a) How many seven-letter words can be formed using the letters of the word "BENZENE"? We seek the number of permutations of seven objects of which three are alike (the three E's), and two are alike (the two N's). By Theorem 8.4, there are

\[
\frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 420
\]
such words.

(b) How many different signals, each consisting of eight flags hung in a vertical line, can be formed from a set of four indistinguishable red flags, three indistinguishable white flags, and a blue flag? We seek the number of permutations of eight objects of which four are alike and three are alike. There are

\[
\frac{8!}{4!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 280
\]
different signals.

8.6 COMBINATIONS

Suppose we have a collection of \( n \) objects. A combination of these \( n \) objects taken \( r \) at a time is any selection of \( r \) of the objects where order doesn't count. In other words, an \( r \)-combination of a set of \( n \) objects is any subset of \( r \) elements. For example, the combinations of the letters \( a, b, c, d \) taken three at a time are

\[
\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \quad \text{or simply} \quad abc, abd, acd, bcd
\]

Observe that the following combinations are equal:

\[
abc, acb, bac, bca, cab \text{ and }eca
\]

That is, each denotes the same set \( \{a, b, c\} \).

The number of combinations of \( n \) objects taken \( r \) at a time is denoted by \( C(n, r) \). The symbols \( ^nC_r \), \( C_{n,r} \) and \( C^n_r \) also appear in various texts. Before we give the general formula for \( C(n, r) \), we consider a special case.

EXAMPLE 8.6. How many combinations are there of the four objects, \( a, b, c \) and \( d \), taken three at a time?

Each combination consisting of three objects determines \( 3! = 6 \) permutations of the objects in the combination.

<table>
<thead>
<tr>
<th>Combinations</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>abc</td>
<td>abc, acb, bac, bca, cab, cba</td>
</tr>
<tr>
<td>abd</td>
<td>abd, adb, bad, bda, dab, dba</td>
</tr>
<tr>
<td>acd</td>
<td>acd, adc, ced, caa, dac, cda</td>
</tr>
<tr>
<td>bcd</td>
<td>bcd, bdc, cdb, dbc, abc, dcb</td>
</tr>
</tbody>
</table>

Thus the number of combinations multiplied by \( 3! \) equals the number of permutations:

\[
C(4, 3) \cdot 3! = P(4, 3) \quad \text{or} \quad C(4, 3) = \frac{P(4, 3)}{3!}
\]

But \( P(4, 3) = 4 \cdot 3 \cdot 2 = 24 \) and \( 3! = 6 \); hence \( C(4, 3) = 4 \) as noted above.

Since any combination of \( n \) objects taken \( r \) at a time determines \( r! \) permutations of the objects in the combination, we can conclude that

\[
P(n, r) = r!C(n, r)
\]

Thus we obtain
Theorem 8.5: \[ C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} \]

Recall that the binomial coefficient \( \binom{n}{r} \) was defined to be \( \frac{n!}{r!(n-r)!} \); hence

\[ C(n, r) = \binom{n}{r} \]

We shall use \( C(n, r) \) and \( \binom{n}{r} \) interchangeably.

**EXAMPLE 8.7.**

(a) How many committees of three can be formed from eight people? Each committee is, essentially, a combination of the eight people taken three at a time. Thus

\[ C(8, 3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56 \]

different committees can be formed.

(b) A farmer buys three cows, two pigs and four hens from a man who has six cows, five pigs and eight hens. How many choices does the farmer have?

The farmer can choose the cows in \( \binom{6}{3} \) ways, the pigs in \( \binom{5}{2} \) ways, and the hens in \( \binom{8}{4} \) ways.

Hence altogether, he can choose the animals in

\[ \binom{6}{3} \binom{5}{2} \binom{8}{4} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 20 \cdot 10 \cdot 70 = 14,000 \text{ ways} \]

**8.7 ORDERED PARTITIONS**

Suppose an urn \( A \) contains seven marbles numbered 1 through 7. We compute the number of ways we can draw, first, two marbles from the urn, then three marbles from the urn, and lastly two marbles from the urn. In other words, we want to compute the number of ordered partitions

\[ (A_1, A_2, A_3) \]

of the set of seven marbles into cells \( A_1 \) containing two marbles, \( A_2 \) containing three marbles and \( A_3 \) containing two marbles. We call these ordered partitions since we distinguish between

\[ \{(1, 2), (3, 4, 5), (6, 7)\} \quad \text{and} \quad \{(6, 7), (3, 4, 5), (1, 2)\} \]

each of which determines the same partition of \( A \).

Now we begin with seven marbles in the urn, so there are \( \binom{7}{2} \) ways of drawing the first two marbles, i.e., of determining the first cell \( A_1 \); following this, there are five marbles left in the urn and so there are \( \binom{5}{3} \) ways of drawing the three marbles, i.e., of determining the second cell \( A_2 \); finally, there are two marbles left in the urn and so there are \( \binom{2}{2} \) ways of determining the last cell \( A_3 \). Hence there are

\[ \binom{7}{2} \binom{5}{3} \binom{2}{2} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2} = 210 \]

different ordered partitions of \( A \) into cells \( A_1 \) containing two marbles, \( A_2 \) containing three marbles, and \( A_3 \) containing two marbles.
Now observe that
\[
\binom{7}{2} \binom{5}{3} \binom{2}{2} = \frac{7!}{2!5!} \cdot \frac{5!}{3!2!} \cdot \frac{2!}{2!1!0!} = \frac{7!}{2!3!2!1!2!}
\]
since each numerator after the first is canceled by the second term in the denominator of the previous factor.

The above discussion can be shown to hold in general. Namely,

**Theorem 8.6:** Let \( A \) contain \( n \) elements and let \( n_1, n_2, \ldots, n_r \) be positive integers with \( n_1 + n_2 + \cdots + n_r = n \). Then there exist

\[
\frac{n!}{n_1! n_2! n_3! \cdots n_r!}
\]

different ordered partitions of \( A \) of the form \((A_1, A_2, \ldots, A_r)\) where \( A_1 \) contains \( n_1 \) elements, \( A_2 \) contains \( n_2 \) elements, \ldots, and \( A_r \) contains \( n_r \) elements.

We apply this theorem in the next example.

**Example 8.6.** In how many ways can nine toys be divided between four children if the youngest child is to receive three toys and each of the others two toys?

We wish to find the number of ordered partitions of the nine toys into four cells containing 3, 2, 2 and 2 toys respectively. By Theorem 8.6, there are

\[
\frac{9!}{3!2!2!2!} = 7560
\]
such ordered partitions.

8.8 **TREE DIAGRAMS**

A (rooted) tree diagram is a useful device to enumerate all the logical possibilities of a sequence of events where each event can occur in a finite number of ways. We illustrate this method with an example.

**Example 8.9.** A man has time to play roulette at most five times. At each play he wins or loses a dollar. The man begins with one dollar and will stop playing before the five times if he loses all his money or if he wins three dollars, i.e., if he has four dollars. The tree diagram describes the way the betting can occur. Each number in the diagram denotes the number of dollars he has at that point.

The betting can occur in eleven different ways. Note that he will stop betting before the five times are up in only three of the cases.
Solved Problems

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

8.1. Compute 4!, 5!, 6!, 7! and 8!

\[ 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \hspace{1cm} 7! = 7 \cdot 6! = 7 \cdot 720 = 5040 \]

\[ 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 24 = 120 \hspace{1cm} 8! = 8 \cdot 7! = 8 \cdot 5040 = 40,320 \]

\[ 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6 \cdot 5! = 6 \cdot 120 = 720 \]

8.2. Compute: (a) \( \frac{13!}{11!} \), (b) \( \frac{7!}{10!} \)

(a) \( \frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11}{11!} = \frac{13 \cdot 12}{11!} = 13 \cdot 12 = 156 \)

(b) \( \frac{7!}{10!} = \frac{7!}{10 \cdot 9 \cdot 8 \cdot 7} = \frac{1}{10 \cdot 9 \cdot 8} = \frac{1}{720} \)

8.3. Write in terms of factorials: (a) 27 \cdot 26, (b) \( \frac{1}{14 \cdot 15 \cdot 12} \)

(a) \( 27 \cdot 26 = \frac{27 \cdot 26}{25!} = \frac{27!}{26!} \)

(b) \( \frac{1}{14 \cdot 15 \cdot 12} = \frac{11!}{14 \cdot 13 \cdot 12 \cdot 11!} = \frac{11!}{14!} \)

8.4. Simplify: (a) \( \frac{n!}{(n-1)!} \), (b) \( \frac{(n+2)!}{n!} \)

(a) \( \frac{n!}{(n-1)!} = \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{(n-1)(n-2) \cdots 2 \cdot 1} = n \) or simply \( \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n \)

(b) \( \frac{(n+2)!}{n!} = \frac{(n+2)(n+1)(n-1)(n-2) \cdots 2 \cdot 1}{n(n-1)(n-2) \cdots 2 \cdot 1} = (n+2)(n+1) = n^2 + 3n + 2 \)

or simply \( \frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2 \)

8.5. Compute: (a) \( \binom{16}{3} \), (b) \( \binom{12}{4} \), (c) \( \binom{15}{5} \)

Recall that there are as many factors in the numerator as in the denominator.

(a) \( \binom{16}{3} = \frac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3} = 560 \)

(b) \( \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495 \)

(c) \( \frac{15}{5} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3003 \)
8.6. Compute: (a) \( \binom{8}{5} \), (b) \( \binom{9}{7} \)

(a) \( \binom{8}{5} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 56 \)

Note that \( 8 - 5 = 3 \); hence we could also compute \( \binom{8}{5} \) as follows:

\[
\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56
\]

(b) Now \( 9 - 7 = 2 \); hence \( \binom{9}{7} = \frac{9 \cdot 8}{1 \cdot 2} = 36 \).

3.7. Prove: \( \binom{17}{6} = \binom{16}{5} + \binom{16}{6} \)

Now \( \binom{16}{5} + \binom{16}{6} = \binom{17}{6} \). Multiply the first fraction by \( \frac{6}{6} \) and the second by \( \frac{11}{11} \) to obtain the same denominator in both fractions; and then add:

\[
\binom{16}{5} + \binom{16}{6} = \frac{6 \cdot 16!}{6! \cdot 11!} + \frac{11 \cdot 16!}{6! \cdot 11! \cdot 10!} = \frac{6 \cdot 16! + 11 \cdot 16!}{6! \cdot 11! \cdot 10!} = \frac{17 \cdot 16!}{6! \cdot 11! \cdot 10!} = \binom{17}{6}
\]

8.8. Prove Theorem 8.1: \( \binom{n + 1}{r} = \binom{n}{r - 1} + \binom{n}{r} \)

(The technique in this proof is similar to that of the preceding problem.)

Now \( \binom{n}{r - 1} + \binom{n}{r} = \frac{n!}{(r - 1)! \cdot (n - r + 1)!} + \frac{n!}{r! \cdot (n - r)!} \). To obtain the same denominator in both fractions, multiply the first fraction by \( \frac{r!}{r!} \) and the second fraction by \( \frac{n - r + 1}{n - r + 1} \). Hence

\[
\binom{n}{r - 1} + \binom{n}{r} = \frac{r \cdot n!}{r! \cdot (n - r + 1)!} + \frac{(n - r + 1) \cdot n!}{(n - r)! \cdot n!} = \frac{r \cdot n! + (n - r + 1) \cdot n!}{r! \cdot (n - r + 1)!} = \frac{r \cdot n!}{r! \cdot (n - r + 1)!} + \frac{(n - r + 1) \cdot n!}{r! \cdot (n - r + 1)!} = \frac{n - r + 1}{r! \cdot (n - r + 1)!} = \binom{n - r + 1}{r}
\]

PERMUTATIONS

8.9. There are four bus lines between \( A \) and \( B \); and three bus lines between \( B \) and \( C \). In how many ways can a man travel (a) by bus from \( A \) to \( C \) by way of \( B \)? (b) roundtrip by bus from \( A \) to \( C \) by way of \( B \)? (c) roundtrip by bus from \( A \) to \( C \) by way of \( B \), if he doesn’t want to use a bus line more than once?

(a) There are four ways to go from \( A \) to \( B \) and three ways to go from \( B \) to \( C \); hence there are \( 4 \cdot 3 = 12 \) ways to go from \( A \) to \( C \) by way of \( B \).

(b) There are twelve ways to go from \( A \) to \( C \) by way of \( B \), and 12 ways to return. Hence there are \( 12 \cdot 12 = 144 \) ways to travel roundtrip.
(c) The man will travel from $A$ to $B$ to $C$ to $B$ to $A$. Enter these letters with connecting arrows as follows:

$$A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$$

The man can travel four ways from $A$ to $B$ and three ways from $B$ to $C$; but he can only travel two ways from $C$ to $B$ and three ways from $B$ to $A$ since he doesn’t want to use a bus line more than once. Enter these numbers above the corresponding arrows as follows:

$$A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$$

Thus there are $4 \cdot 3 \cdot 2 \cdot 3 = 72$ ways to travel roundtrip without using the same bus line more than once.

8.10. Suppose repetitions are not permitted. (a) How many three-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9? (b) How many of these numbers are less than 400? (c) How many are even? (d) How many are odd? (e) How many are multiples of 5?

In each case draw three boxes $\boxed{} \boxed{} \boxed{}$ to represent an arbitrary number, and then write in each box the number of digits that can be placed there.

(a) The box on the left can be filled in six ways; following this, the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways: $6 \boxed{} \boxed{}$. Thus there are $6 \cdot 5 \cdot 4 = 120$ numbers.

(b) The box on the left can be filled in only two ways, by 2 or 3, since each number must be less than 400; the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways: $2 \boxed{} \boxed{}$. Thus there are $2 \cdot 5 \cdot 4 = 40$ numbers.

(c) The box on the right can be filled in only two ways, by 2 or 6, since the numbers must be even; the box on the left can then be filled in five ways; and, lastly, the middle box can be filled in four ways: $\boxed{} 5 \boxed{}$. Thus there are $5 \cdot 4 \cdot 2 = 40$ numbers.

(d) The box on the right can be filled in only four ways, by 3, 5, 7 or 9, since the numbers must be odd; the box on the left can then be filled in five ways; and, lastly, the box in the middle can be filled in four ways: $\boxed{} 5 \boxed{}$. Thus there are $5 \cdot 4 \cdot 4 = 80$ numbers.

(e) The box on the right can be filled in only one way, by 5, since the numbers must be multiples of 5; the box on the left can then be filled in five ways; and, lastly, the box in the middle can be filled in four ways: $\boxed{} 5 \boxed{}$. Thus there are $5 \cdot 4 \cdot 1 = 20$ numbers.

8.11. In how many ways can a party of seven persons arrange themselves (a) in a row of seven chairs? (b) around a circular table?

(a) The seven persons can arrange themselves in a row in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$ ways.

(b) One person can sit at any place in the circular table. The other six persons can then arrange themselves in $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways around the table.

This is an example of a circular permutation. In general, $n$ objects can be arranged in a circle in $(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = (n-1)!$ ways.

8.12. (a) In how many ways can three boys and two girls sit in a row? (b) In how many ways can they sit in a row if the boys and girls are each to sit together? (c) In how many ways can they sit in a row if just the girls are to sit together?

(a) The five persons can sit in a row in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways.
(b) There are two ways to distribute them according to sex: BBBGG or GGBBR. In each case the boys can sit in \[3 \times 2 \times 1 = 6\] ways, and the girls can sit in \[2 \times 1 = 2\] ways. Thus, altogether, there are \[2 \times 3! \times 2! = 2 \times 6 \times 2 = 24\] ways.

(c) There are four ways to distribute them according to sex: GBBB, BBGB, BBGB, BBBB. Note that each way corresponds to the number, 0, 1, 2 or 3, of boys sitting to the left of the girls. In each case, the boys can sit in \[3!\] ways, and the girls in \[2!\] ways. Thus, altogether, there are \[4 \times 3! \times 2! = 4 \times 6 \times 2 = 48\] ways.

8.13. Solve the preceding problem in the case of \(r\) boys and \(s\) girls. (Answers are to be left in factorials.)

(a) The \(r + s\) persons can sit in a row in \((r + s)!\) ways.

(b) There are still two ways to distribute them according to sex, the boys on the left or the girls on the left. In each case the boys can sit in \(r!\) ways and the girls in \(s!\) ways. Thus, altogether, there are \(2 \times r! \times s!\) ways.

(c) There are \(r + 1\) ways to distribute them according to sex, each corresponding to the number, 0, 1, \ldots, \(r\), of boys sitting to the left of the girls. In each case the boys can sit in \(r!\) ways and the girls in \(s!\) ways. Thus, altogether, there are \((r + 1) \times r! \times s!\) ways.

8.14. Find the number of distinct permutations that can be formed from all the letters of each word: (a) THEM, (b) THAT, (c) RADAR, (d) UNUSUAL, (e) SOCIOLOGICAL.

(a) \[4! = 24,\] since there are four letters and no repetition.

(b) \[\frac{4!}{2!} = 12,\] since there are four letters of which two are T.

(c) \[\frac{5!}{2!2!} = 30,\] since there are five letters of which two are R and two are A.

(d) \[\frac{7!}{3!} = 840,\] since there are seven letters of which three are U.

(e) \[\frac{12!}{3!2!2!1!} = \text{since there are twelve letters of which three are O, two are C, two are L, and two are E},\]

8.15. In how many ways can four mathematics books, three history books, three chemistry books and two sociology books be arranged on a shelf so that all books of the same subject are together?

First the books must be arranged on the shelf in four units according to subject matter: \[\square \square \square \square \]. The box on the left can be filled by any of the four subjects; the next by any three subjects remaining; the next by any two subjects remaining; and the box on the right by the last subject: \[\begin{array}{c} 4 \ 3 \ 2 \ 1 \end{array} \]. Thus there are \[4 \times 3 \times 2 \times 1 = 4!\] ways to arrange the books on the shelf according to subject matter.

In each of the above cases, the mathematics books can be arranged in \(4!\) ways, the history books in \(3!\) ways, the chemistry books in \(3!\) ways, and the sociology books in \(2!\) ways. Thus, altogether, there are \(4! \times 3! \times 3! \times 2! = 41472\) arrangements.

8.16. Find the total number of positive integers that can be formed from the digits 1, 2, 3 and 4 if no digit is repeated in any one integer.

Note that no integer can contain more than four digits. Let \(a_1, a_2, a_3\) and \(a_4\) denote the number of integers containing 1, 2, 3 and 4 digits respectively. We compute each \(a_i\) separately.
Since there are four digits, there are four integers containing exactly one digit, i.e. \( s_1 = 4 \).
Also, since there are four digits, there are \( 4 \cdot 3 = 12 \) integers containing two digits, i.e. \( s_2 = 12 \).
Similarly, there are \( 4 \cdot 3 \cdot 2 = 24 \) integers containing three digits and \( 4 \cdot 3 \cdot 2 \cdot 1 = 24 \) integers containing four digits, i.e. \( s_3 = 24 \) and \( s_4 = 24 \). Thus, altogether, there are \( s_1 + s_2 + s_3 + s_4 = 4 + 12 + 24 + 24 = 64 \) integers.

8.17. Find \( n \) if 
(a) \( P(n, 2) = 72 \),
(b) \( P(n, 4) = 42P(n, 2) \),
(c) \( 2P(n, 2) + 50 = P(2n, 2) \).

(a) \( P(n, 2) = n(n-1) = n^2 - n \); hence \( n^2 - n = 72 \) or \( n^2 - n - 72 = 0 \) or \( (n-9)(n+8) = 0 \).

Since \( n \) must be positive, the only answer is \( n = 9 \).

(b) \( P(n, 4) = n(n-1)(n-2)(n-3) \) and \( P(n, 2) = n(n-1) \). Hence
\[
\frac{n(n-1)(n-2)(n-3)}{n(n-1)} = 42n(n-1) \quad \text{or, if } n \neq 0, n \neq 1, \quad n^2 - 5n + 6 = 42 \quad \text{or} \quad n^2 - 5n - 36 = 0 \quad \text{or} \quad (n-9)(n+4) = 0
\]
Since \( n \) must be positive, the only answer is \( n = 9 \).

(c) \( P(n, 2) = n(n-1) = n^2 - n \) and \( P(2n, 2) = 2n(2n-1) = 4n^2 - 2n \). Hence
\[
2(n^2 - n) + 50 = 4n^2 - 2n \quad \text{or} \quad 2n^2 - 2n + 50 = 4n^2 - 2n \quad \text{or} \quad 50 = 2n^2 \quad \text{or} \quad n^2 = 25
\]
Since \( n \) must be positive, the only answer is \( n = 5 \).

COMBINATIONS

8.18. In how many ways can a committee consisting of three men and two women be chosen from seven men and five women?

The three men can be chosen from the seven men in \( \binom{7}{3} \) ways, and the two women can be chosen from the five women in \( \binom{5}{2} \) ways. Hence the committee can be chosen in
\[
\binom{7}{3} \cdot \frac{5}{2} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{5 \cdot 4}{1 \cdot 2} = 350 \text{ ways}
\]

8.19. A bag contains six white marbles and five black marbles. Find the number of ways four marbles can be drawn from the bag if (a) they can be any color, (b) two must be white and two black, (c) they must all be of the same color.

(a) The four marbles (of any color) can be chosen from the eleven marbles in
\[
\binom{11}{4} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 330 \text{ ways}
\]

(b) The two white marbles can be chosen in \( \binom{6}{2} \) ways, and the two black marbles can be chosen in \( \binom{5}{2} \) ways. Thus there are
\[
\binom{6}{2} \cdot \binom{5}{2} = \frac{6 \cdot 5}{1 \cdot 2} \cdot \frac{5 \cdot 4}{1 \cdot 2} = 150 \text{ ways of drawing two white marbles and two black marbles.}
\]

(c) There are \( \binom{6}{4} = 15 \) ways of drawing four white marbles, and \( \binom{5}{4} = 5 \) ways of drawing four black marbles. Thus there are \( 15 + 5 = 20 \) ways of drawing four marbles of the same color.

8.20. A delegation of four students is selected each year from a college to attend the National Student Association annual meeting. (a) In how many ways can the delegation be chosen if there are twelve eligible students? (b) In how many ways if two of the eligible students will not attend the meeting together? (c) In how many ways if two of the eligible students are married and will only attend the meeting together?
(a) The four students can be chosen from the twelve students in \( \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495 \) ways.

(b) Let \( A \) and \( B \) denote the students who will not attend the meeting together.

Method 1.
If neither \( A \) nor \( B \) is included, then the delegation can be chosen in \( \binom{10}{3} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210 \) ways. If either \( A \) or \( B \), but not both, is included, then the delegation can be chosen in \( 2 \cdot \binom{10}{3} = 2 \cdot \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 240 \) ways. Thus, altogether, the delegation can be chosen in \( 210 + 240 = 450 \) ways.

Method 2.
If \( A \) and \( B \) are both included, then the other two members of the delegation can be chosen in \( \binom{10}{2} = 45 \) ways. Thus there are \( 495 - 45 = 450 \) ways the delegation can be chosen if \( A \) and \( B \) are not both included.

(c) Let \( C \) and \( D \) denote the married students. If \( C \) and \( D \) do not go, then the delegation can be chosen in \( \binom{10}{4} = 210 \) ways. If both \( C \) and \( D \) go, then the delegation can be chosen in \( \binom{10}{2} = 45 \) ways. Altogether, the delegation can be chosen in \( 210 + 45 = 255 \) ways.

A student is to answer eight out of ten questions on an exam. (a) How many choices has he? (b) How many if he must answer the first three questions? (c) How many if he must answer at least four of the first five questions?

\[ \binom{10}{8} = \frac{10!}{8!2!} = \frac{10 \cdot 9}{2} = 45 \] ways.

(b) If he answers the first three questions, then he can choose the other five questions from the last seven questions in \( \binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21 \) ways.

(c) If he answers all the first five questions, then he can choose the other three questions from the last five in \( \binom{5}{3} = 10 \) ways. On the other hand, if he answers only four of the first five questions, then he can choose these four in \( \binom{5}{4} = \binom{5}{1} = 5 \) ways, and he can choose the other four questions from the last five in \( \binom{5}{4} = \binom{5}{1} = 5 \) ways; hence he can choose the eight questions in \( 5 \cdot 5 = 25 \) ways. Thus he has a total of thirty-five choices.

There are twelve points \( A, B, \ldots \) in a given plane, no three on the same line. (a) How many lines are determined by the points? (b) How many lines pass through the point \( A \)? (c) How many triangles are determined by the points? (d) How many of these triangles contain the point \( A \) as a vertex?

(a) Since two points determine a line, there are \( \binom{12}{2} = \frac{12 \cdot 11}{1 \cdot 2} = 66 \) lines.

(b) To determine a line through \( A \), one other point must be chosen; hence there are eleven lines through \( A \).

(c) Since three points determine a triangle, there are \( \binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220 \) triangles.
(d) Method 1.

To determine a triangle with vertex \( A \), two other points must be chosen; hence there are

\[
\binom{11}{2} = \frac{11 \cdot 10}{1 \cdot 2} = 55 \text{ triangles with } A \text{ as a vertex.}
\]

Method 2.

There are \( \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165 \) triangles without \( A \) as a vertex. Thus \( 220 - 165 = 55 \) of the triangles do have \( A \) as a vertex.

8.23. How many committees of five with a given chairman can be selected from twelve persons?

The chairman can be chosen in twelve ways and, following this, the other four on the committee can be chosen from the eleven remaining in \( \binom{11}{4} \) ways. Thus there are \( 12 \cdot \binom{11}{4} = 12 \cdot 330 = 3960 \) such committees.

8.24. Find the number of subsets of a set \( X \) containing \( n \) elements.

Method 1.

The number of subsets of \( X \) with \( r \equiv n \) elements is given by \( \binom{n}{r} \). Hence, altogether, there are

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n - 1} + \binom{n}{n}
\]

subsets of \( X \). The above sum is equal to \( 2^n \) (Problem 8.41), and so there are \( 2^n \) subsets of \( X \).

Method 2.

There are two possibilities for each element of \( X \): either it belongs to the subset or it doesn’t; hence there are

\[
\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n
\]

ways to form a subset of \( X \), i.e. there are \( 2^n \) different subsets of \( X \).

8.25. In how many ways can a teacher choose one or more students from six eligible students?

Method 1.

By the preceding problem, there are \( 2^6 = 64 \) subsets of the set consisting of the six students. However, the empty set must be deleted since one or more students are chosen. Accordingly, there are \( 2^6 - 1 = 64 - 1 = 63 \) ways to choose the students.

Method 2.

Either 1, 2, 3, 4, 5 or 6 students are chosen. Hence the number of choices is

\[
\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 6 + 15 + 20 + 15 + 6 + 1 = 63
\]

8.26. In how many ways can three or more persons be selected from twelve persons?

There are \( 2^{12} - 1 = 4095 - 1 = 4094 \) ways of choosing one or more of the twelve persons.

Now there are \( \binom{12}{1} + \binom{12}{2} = 12 + 66 = 78 \) ways of choosing one or two of the twelve persons.

Hence there are \( 4095 - 78 = 4017 \) ways of choosing three or more.